▸ DARÍO GARCÍA, *Model Theory of Pseudofinite Structures.*
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A structure M is said to be *pseudofinite* if every first-order sentence that is true in M has a finite model, or equivalently, if M is elementarily equivalent to an ultraproduct of finite structures. For this kind of ultraproducts, the fundamental theorem of ultraproducts (Łoś' Theorem) provides a powerful connection between finite and infinite structures, which can be used to prove qualitative properties of large finite structures using combinatorial methods applied to non-standard cardinalities of definable sets. Also, in the other direction, quantitative properties in classes of finite structures often induce desirable model-theoretic properties in their ultraproducts.

The idea is that the counting measure on a class of finite structures can be lifted using Łoś' theorem to give notions of dimension and measure on their ultraproduct. This allows ideas from geometric model theory to be used in the context of pseudofinite theories, and potentially we can prove results in finite combinatorics (of graphs, groups, fields, etc) by studying the corresponding properties in the ultraproducts.

This approach was used by Hrushovski and Wagner in [10], but was better explored in his striking papers [8] and [9], where he applies ideas from geometric model theory to additive combinatorics, locally compact groups and linear approximate subgroups. On the other hand, Goldbring and Towsner developed in [6] the Approximate Measure Logic, a logical framework that serves as a formalization of connections between finitary combinatorics and diagonalization arguments in measure theory or ergodic theory that have appeared in various places throughout the literature. Using this, Goldbring and Towsner gave proofs of the Furstenberg's correspondence principle, Szemerédi's Regularity Lemma, the triangle removal lemma, and Szemerédi's Theorem: every subset of the integers with positive density contains arbitrarily long arithmetic progressions.

More recently there has been an increasing interest in applications of model-theoretic properties to combinatorics, starting with the Regularity Lemma for stable graphs due to Malliaris-Shelah (see [12]) and including several versions of the regularity lemma in different contexts: the algebraic regularity lemma for sufficiently large fields [14], regularity lemmas in distal structures ([3]) and the stable regularity lemma for groups (see [15], [4]).

In the other direction, we have the concept of asymptotic classes of finite structures, defined by Macpherson and Steinhorn in [11] as classes of finite structures that satisfy strong conditions on the sizes of definable sets. The most notable examples are the class of finite fields, the class of cyclic groups, or the class of Paley graphs. The infinite ultraproducts of asymptotic classes are all supersimple of finite SU-rank, but recent generalizations of this concept (known as *multidimensional asymptotic classes*, or m.a.c.) are more flexible and allow the presence of ultraproducts whose SU-rank is possibly infinite. (cf. [1], [16]).

In this series of lectures I will recall some model-theoretic results regarding ultraproducts of finite structures, and review some of the applications that can be derived from them. On the model-theoretic side, we will study the so-called "pseudofinite dimension" (cf. [9]) and its relationship with the forking and model-theoretic dividing lines in pseudofinite structures as presented in [5]. We will also introduce the concept of multidimensional asymptotic classes having as motivating example the theory of the *everywhere infinite forest*, which is the theory of an acyclic graph G such that every vertex has infinite degree, a well-known example of an ω-stable theory of infinite rank.

On the applications to combinatorics we will see a proof of Szemerédi's Regularity Lemma using ultraproducts of finite structures, (cf. [6]) as well as some improvements of this result under the assumption of stability due to Malliaris and Shelah. This last approach was used by Chernikov and Starchenko in [2] to obtain some progress in the study of the Erdős-Hajnal conjecture.

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